by Theorem 3.10, $(y_n)_{n\in\mathbb{N}}$ (and hence $(x_n)_{n\in\mathbb{N}}$) has a subsequence $(z_n)_{n\in\mathbb{N}}$ which converges to some $M \in \mathbb{R}$. Since $|z_n - L| \geq \varepsilon\ \forall n \in \mathbb{N}$, $M \neq L$. Thus, M and L are 2 distinct subsequential limits of $(x_n)_{n\in\mathbb{N}}$.

4. $\mathbb{R}$ since $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$ (Exercise 2.3.4).

5. $\{\frac{1}{n}\}_{n\in\mathbb{N}} \cup \{1 + \frac{1}{n}\}_{n\in\mathbb{N}} \cup \{2 + \frac{1}{n}\}_{n\in\mathbb{N}}$ has 0,1, and 2 as accumulation points.

6. We have $A \subset \mathbb{R}$ and $x \in \mathbb{R}$. Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence of distinct points in A with $x_n \to x$, and let U be a neighborhood of x. Since $(x_n)_{n\in\mathbb{N}}$ is eventually in U, $U \cap A$ is infinite, and so x is an accumulation point of A by Proposition 3.4.
   Now suppose that x is an accumulation point of A. By Proposition 3.4, $\left(x - \dfrac{1}{n}, x + \dfrac{1}{n}\right) \cap A$ is infinite $\forall n \in \mathbb{N}$. Choose $x_1 \in (x-1, x+1) \cap A$; choose $x_2 \in \left(x - \dfrac{1}{2}, x + \dfrac{1}{2}\right) \cap A$ with $x_2 \neq x_1$; choose $x_3 \in \left(x - \dfrac{1}{3}, x + \dfrac{1}{3}\right) \cap A$ with $x_3 \notin \{x_1, x_2\}$; etc. Then $(x_n)_{n\in\mathbb{N}}$ is a sequence of distinct points in A and $x_n \to x$.

7. Since x is an isolated point of A, $\exists$ a neighborhood V of x such that $V \cap A = \{x\}$. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in A which converges to x, then $(x_n)_{n\in\mathbb{N}}$ must eventually be in $V \cap A$. Therefore, $(x_n)_{n\in\mathbb{N}}$ must eventually be the constant x.

8. Let $(x_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{Q}$ which converges to $\sqrt{2}$. (Such a sequence exists since $\mathbb{Q}$ is dense in $\mathbb{R}$.) Then $\{x_n : n \in \mathbb{N}\}$ is a bounded infinite subset of $\mathbb{Q}$ whose only accumulation point is $\sqrt{2}$.

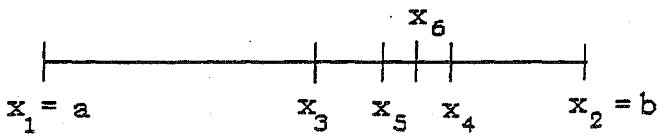
3.6 Cauchy Sequences

1. (a) $\left(\frac{1}{n}\right)_{n=2}^{\infty}$, or any sequence in $(0,1)$ which converges to 0 or 1.
   (b) By Theorem 3.12, a Cauchy sequence in $[0,1]$ must converge to a real number, say x. By Theorem 3.3, $x \in [0,1]$.

2. (a) By Exercise 3.4.4, $(x_n)_{n\in\mathbb{N}}$ does not converge. By Theorem 3.12, $(x_n)_{n\in\mathbb{N}}$ is not Cauchy.
   (b) $|x_{n+1} - x_n| = \dfrac{1}{n+1} \to 0$.

3. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{Z}$, and let $\varepsilon = 1$. By Definition 3.11, $\exists\, n_0 \in \mathbb{N}$ such that $n, m \geq n_0 \Rightarrow |x_n - x_m| < 1$. Since x_n and x_m are integers, $n, m \geq n_0 \Rightarrow x_n - x_m = 0$ or $x_n = x_m$. Hence, $(x_n)_{n\in\mathbb{N}}$ is eventually constant.
Since an eventually constant sequence is Cauchy, a sequence in $\mathbb{Z}$ converges $\iff$ it is Cauchy (since $\mathbb{Z} \subset \mathbb{R}$) $\iff$ it is eventually constant. Thus, a convergent sequence in $\mathbb{Z}$ must converge to an integer.

4. (a) Let $L = b - a$.



(b) $|x_2 - x_1| = L = \dfrac{L}{2^0}$. Let $k \geq 1$ and assume $|x_{k+1} - x_k| = \dfrac{L}{2^{k-1}}$.
Then

$$\begin{aligned}
|x_{k+2} - x_{k+1}| &= \left|\frac{x_{k+1} + x_k}{2} - x_{k+1}\right| \\
&= \frac{1}{2}|x_k - x_{k+1}| \\
&= \frac{1}{2} \cdot \frac{L}{2^{k-1}} = \frac{L}{2^k}.
\end{aligned}$$

By induction, $|x_{n+1} - x_n| = \dfrac{L}{2^{n-1}}\ \forall\, n \in \mathbb{N}$.
(c) For $m > n$,

$$\begin{aligned}
|x_m - x_n| &\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\
&= \frac{L}{2^{m-2}} + \frac{L}{2^{m-3}} + \cdots + \frac{L}{2^{n-1}} \\
&= \frac{L}{2^{n-1}}\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^{m-n-1}\right) \\
&= \frac{L}{2^{n-1}} \cdot \frac{1 - \left(\frac{1}{2}\right)^{m-n}}{1 - \frac{1}{2}} \\
&= \frac{L}{2^{n-2}}\left(1 - \left(\frac{1}{2}\right)^{m-n}\right) \\
&< \frac{L}{2^{n-2}}.
\end{aligned}$$

(d) Let $\varepsilon > 0$. Since $\frac{L}{2^{n-2}} \to 0, \exists\, n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow \frac{L}{2^{n-2}} < \varepsilon$. Let $n, m \geq n_0$. If $m > n$, then $|x_m - x_n| < \frac{L}{2^{n-2}} < \varepsilon$; while if $n > m$, then $|x_m - x_n| < \frac{L}{2^{m-2}} < \varepsilon$. Therefore, $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

5. (a) From Exercise 4(b), $|x_{2k+2} - x_{2k+1}| = \frac{L}{2^{2k}}$. From the picture in Exercise 4(a), it is clear that $x_{2k+2} > x_{2k+1}\ \forall\, k$, and so we can remove the absolute value. To show this by induction, assume $x_{2k+2} > x_{2k+1}$. Then

$$\begin{aligned}
x_{2k+4} - x_{2k+3} &= \frac{x_{2k+3} + x_{2k+2}}{2} - x_{2k+3} \\
&= \frac{x_{2k+2} - x_{2k+3}}{2} \\
&= \frac{x_{2k+2} - \dfrac{x_{2k+2} + x_{2k+1}}{2}}{2} \\
&= \frac{x_{2k+2} - x_{2k+1}}{4} > 0
\end{aligned}$$

by the induction hypothesis.

(b) For $n = 1$, $x_3 = \frac{x_1 + x_2}{2} = \frac{a+b}{2} = \frac{a+a+L}{2} = a + \frac{L}{2}$. Let $k \geq 1$ and assume $x_{2k+1} = a + \frac{L}{2} + \frac{L}{2^3} + \cdots + \frac{L}{2^{2k-1}}$. Then

$$\begin{aligned}
x_{2k+3} &= \frac{x_{2k+2} + x_{2k+1}}{2} = \frac{x_{2k+1} + \frac{L}{2^{2k}} + x_{2k+1}}{2} \quad \text{(by (a))} \\
&= x_{2k+1} + \frac{L}{2^{2k+1}} \\
&= a + \frac{L}{2} + \frac{L}{2^3} + \cdots + \frac{L}{2^{2k-1}} + \frac{L}{2^{2k+1}}
\end{aligned}$$

by the induction hypothesis. By induction, x_{2n+1} has the desired form $\forall\, n \geq 1$.

(c) $x = \lim_{n\to\infty} x_{2n+1}$ since $(x_{2n+1})_{n\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$. From (b)

$$\begin{aligned}
x_{2n+1} &= a + \frac{L}{2} + \frac{L}{2^3} + \cdots + \frac{L}{2^{2n-1}} \\
&= a + \frac{L}{2}\left(1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots + \left(\frac{1}{4}\right)^{n-1}\right)
\end{aligned}$$

$$\begin{aligned}
&= a+\frac{L}{2}\cdot\frac{1-\left(\frac{1}{4}\right)^n}{1-\frac{1}{4}}\\
&= a+\frac{2}{3}L\left(1-\left(\frac{1}{4}\right)^n\right)\\
&\to a+\frac{2}{3}L\\
&= a+\frac{2}{3}(b-a)=\frac{1}{3}a+\frac{2}{3}b.
\end{aligned}$$

6. First note that

$$\begin{aligned}
|x_{n+2}-x_{n+1}| &\le r\,|x_{n+1}-x_n|\\
&\le r^2\,|x_n-x_{n-1}|\\
&\le r^3\,|x_{n-1}-x_{n-2}|\\
&\le \cdots \le r^n\,|x_2-x_1|.
\end{aligned}$$

Note that the exponent on r is $n+1$ minus the smaller subscript to its right. (This also follows by induction.)
For $m>n$,

$$\begin{aligned}
|x_m-x_n| &\le |x_m-x_{m-1}|+|x_{m-1}-x_{m-2}|+\cdots+|x_{n+1}-x_n|\\
&\le \left(r^{m-2}+r^{m-3}+\cdots+r^{n-1}\right)|x_2-x_1|\\
&= r^{n-1}\left(1+r+r^2+\cdots+r^{m-n-1}\right)|x_2-x_1|\\
&= r^{n-1}\left(\frac{1-r^{m-n}}{1-r}\right)|x_2-x_1|\\
&\le \frac{r^{n-1}}{1-r}|x_2-x_1| \xrightarrow[n]{} 0.
\end{aligned}$$

As in Example 3.23 or Exercise 4(d), it follows that $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

7. First note that $x_n>0\ \forall\, n\in\mathbb{N}$.

(a) $|x_{n+2}-x_{n+1}| = \dfrac{|x_{n+1}-x_n|}{(2+x_{n+1})(2+x_n)} \le \dfrac{1}{4}|x_{n+1}-x_n|$. Thus, the r of Exercise 6 is $\dfrac{1}{4}$ and so $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

(b) By Theorem 3.12, $x_n \to x \in \mathbb{R}$. Note that $x \geq 0$ since each $x_n > 0$. Then

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2+x_n} = \frac{1}{2+x},$$

and so $x^2+2x-1=0$ or $x = -1 \pm \sqrt{2}$ by the quadratic formula. Since $x \geq 0$, $x = -1+\sqrt{2}$.

3.7 Limits at Infinity

1. Let $\alpha > 0$ and $\beta < 0$.

   (a) Since $\mathbb{N}$ is unbounded above, $\exists\, n_0 \in \mathbb{N}$ such that $n_0 > \alpha^2$. Then $n \geq n_0 \Rightarrow \sqrt{n} \geq \sqrt{n_0} > \alpha$.

   (b) Choose $n_0 \in \mathbb{N}$ with $n_0 > \alpha^2$. Then $n \geq n_0 \Rightarrow$

   $$\frac{\sqrt{n^2+1}}{\sqrt{n}} = \sqrt{n+\frac{1}{n}} > \sqrt{n} \geq \sqrt{n_0} > \alpha.$$

   (c) Choose $n_0 \in \mathbb{N}$ with $n_0 > 6+\alpha$. Then $n \geq n_0 \Rightarrow$

   $$n^2-6n+1 > n(n-6) \geq n-6 \geq n_0-6 > \alpha.$$

   (d) Choose $n_0 \in \mathbb{N}$ such that $n_0 > (6+\alpha)^2$. Then $n \geq n_0 \Rightarrow$

   $$n - 6\sqrt{n} = \sqrt{n}\left(\sqrt{n}-6\right) \geq \sqrt{n}-6 \geq \sqrt{n_0}-6 > 6+\alpha-6 = \alpha.$$

   (e) Choose $n_0 \in \mathbb{N}$ with $n_0 > 7+\beta^2$. Then $n \geq n_0 \Rightarrow$

   $$\sqrt{n-7} \geq \sqrt{n_0-7} > \sqrt{\beta^2} = -\beta$$

   and so $-\sqrt{n-7} < \beta$.

   (f) Choose $n_0 \in \mathbb{N}$ with $n_0 > -\beta+1$. Then $n \geq n_0 \Rightarrow n-1 \geq n_0-1 > -\beta$ and so $-n+\sin n \leq -n+1 = -(n-1) < \beta$.

2. (a) Let $x_n \to \infty$ and let $\beta < 0$. Choose $n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow x_n > -\beta$. Then $n \geq n_0 \Rightarrow -x_n < \beta$, and so $-x_n \to -\infty$. Now let $-x_n \to -\infty$ and let $\alpha > 0$. Choose $n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow -x_n < -\alpha$. Then $n \geq n_0 \Rightarrow x_n > \alpha$, and so $x_n \to \infty$. (This also follows from Theorem 3.14, part 4, by using a constant sequence of -1.)

(b) If $x_n \to 0$ and $\alpha > 0$, choose $n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow x_n < \frac{1}{\alpha}$. Then $n \geq n_0 \Rightarrow \frac{1}{x_n} > \alpha$ and so $\frac{1}{x_n} \to \infty$. If $\frac{1}{x_n} \to \infty$ and $\varepsilon > 0$, choose $n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow \frac{1}{x_n} > \frac{1}{\varepsilon}$. Then $n \geq n_0 \Rightarrow x_n < \varepsilon$ and so $x_n \to 0$.

3. (a) If $x_n \to -\infty$ and $\alpha > 0$, then $(x_n)_{n\in\mathbb{N}}$ is eventually in $(-\infty, -\alpha)$. Hence, $(|x_n|)_{n\in\mathbb{N}}$ is eventually in (α, ∞) and so $|x_n| \to \infty$. If $|x_n| \to \infty$ and $\beta < 0$, then $(|x_n|)_{n\in\mathbb{N}}$ is eventually in $(-\beta, \infty)$. Since $|x_n| = -x_n$ $\forall n$, $(x_n)_{n\in\mathbb{N}}$ is eventually in $(-\infty, \beta)$.

   (b) Let $(x_n)_{n\in\mathbb{N}} = (1, -2, 3, -4, 5, -6, \ldots) = \left((-1)^{n+1} n\right)_{n\in\mathbb{N}}$.

4. Proof of 1. Let $\alpha > 0$. Case 1: $x \in \mathbb{R}$. Since $x_n \to x$, $\exists\, n_1 \in \mathbb{N}$ such that $n \geq n_1 \Rightarrow x_n > x - 1$. Since $y_n \to \infty$, $\exists\, n_2 \in \mathbb{N}$ such that $n \geq n_2 \Rightarrow y_n > \alpha - (x-1)$. Then $n \geq \max\{n_1, n_2\} \Rightarrow x_n + y_n > (x-1) + \alpha - (x-1) = \alpha$, and so $x_n + y_n \to \infty$.
   Case 2: $x = \infty$. Since $x_n \to \infty$ and $y_n \to \infty$, $\exists\, n_3$ and $n_4 \in \mathbb{N}$ such that $n \geq n_3 \Rightarrow x_n > \frac{\alpha}{2}$ and $n \geq n_4 \Rightarrow y_n > \frac{\alpha}{2}$. Then $n \geq \max\{n_3, n_4\} \Rightarrow x_n + y_n > \alpha$, and so $x_n + y_n \to \infty$.
   Proof of 2. Let $\beta < 0$. If $x \in \mathbb{R}$, then eventually $x_n < x + 1$ and $z_n < \beta - (x+1)$. So eventually, $x_n + z_n < (x+1) + \beta - (x+1) = \beta$. If $x = -\infty$, then eventually $x_n < \frac{\beta}{2}$ and $z_n < \frac{\beta}{2}$. So eventually, $x_n + z_n < \beta$.
   Proof of 4. By Theorem 3.2 and Exercise 2(a), $-x_n \to -x$ where $0 < -x \leq \infty$. By part 3 of Theorem 3.14, $(-x_n)y_n \to \infty$ and $(-x_n)z_n \to -\infty$. By Exercise 2(a), $x_n y_n \to -\infty$ and $x_n z_n \to \infty$.

5. Since $|x_n| \to \infty$, we may assume $x_n \neq 0$ $\forall n \in \mathbb{N}$. Then $\frac{1}{|x_n|} \to 0$ (Exercise 2(b)). Since $x_n y_n \to L \in \mathbb{R}$, $|y_n| = \frac{1}{|x_n|}|x_n y_n| \to 0 \cdot |L| = 0$ by Theorem 3.2.

6. Since $\frac{x_n}{y_n} \to L$, $\exists\, n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow \frac{L}{2} < \frac{x_n}{y_n} < \frac{3}{2}L$. Hence, $n \geq n_0 \Rightarrow \frac{1}{2}Ly_n < x_n < \frac{3}{2}Ly_n$. If $x_n \to \infty$, then $\frac{3}{2}Ly_n \to \infty$; and since $\frac{3}{2}L > 0$, $y_n = \frac{2}{3L}\left(\frac{3}{2}Ly_n\right) \to \infty$. If $y_n \to \infty$, then $\frac{1}{2}Ly_n \to \infty$

since $\frac{1}{2}L > 0$, and so $x_n \to \infty$.

7. (a) Eventually, $\frac{x_n}{y_n} < 1$ or $x_n < y_n$.

   (b) If $y_n \le B \;\; \forall n \in \mathbb{N}$, then $0 < x_n = y_n \left(\frac{x_n}{y_n}\right) \le B \left(\frac{x_n}{y_n}\right) \to 0$.

8. The subsequences constructed in Exercise 3.3.5 have the appropriate limits.

9. Since $\lim_{n\to\infty} x_n \neq \infty$ and $x_n > 0 \; \forall n$, $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(y_n)_{n\in\mathbb{N}}$ which is bounded by Proposition 3.6. By Theorem 3.10 $(y_n)_{n\in\mathbb{N}}$, and hence $(x_n)_{n\in\mathbb{N}}$, has a convergent subsequence.

10. For $\alpha \in \mathbb{R}$ or $\beta \in \mathbb{R}$ the result follows from Exercise 3.4.9. If $\alpha = \infty$, then A is unbounded above. So $\forall n \in \mathbb{N}$, choose $x_n \in A$ with $x_n > n$. Then $(x_n)_{n\in\mathbb{N}}$ is a sequence in A and $x_n \to \infty$. By Theorem 3.9, $(x_n)_{n\in\mathbb{N}}$ has a monotone subsequence $(y_n)_{n\in\mathbb{N}}$ which has limit ∞ by Theorem 3.15, and which must be monotone increasing for otherwise $\lim_{n\to\infty} y_n \le y_1 < \infty$.
    If $\beta = -\infty$, then A is unbounded below. Hence, $\forall\, n \in \mathbb{N}\; \exists\, x_n \in A$ with $x_n < -n$. Therefore, $(x_n)_{n\in\mathbb{N}}$ is a sequence in A, $x_n \to -\infty$, and as above $(x_n)_{n\in\mathbb{N}}$ has a monotone subsequence $(y_n)_{n\in\mathbb{N}}$ with limit $-\infty$. This subsequence must be monotone decreasing for otherwise $\lim_{n\to\infty} y_n \ge y_1 > -\infty$.

11. Let $\alpha = \sup A$.
    Case 1: $\alpha \in A$. Let $x_n = \alpha \; \forall n$. If $\alpha = -\infty$, then $A = \{-\infty\}$ and so $\alpha = -\infty$ falls into this case.
    Case 2: $\alpha \in \mathbb{R} \setminus A$. Then $A \cap \mathbb{R}$ is nonempty and bounded above, so this follows from Exercise 3.4.9.
    Case 3: $\alpha = \infty \notin A$. This follows from Exercise 10 applied to $A \cap \mathbb{R}$.
    Let $\beta = \inf A$.
    Case 1: $\beta \in A$. Let $x_n = \beta \; \forall n$. Note that if $\beta = +\infty$, then $A = \{+\infty\}$ and so $\beta = +\infty$ falls into this case.
    Case 2: $\beta \in \mathbb{R} \setminus A$. Then $A \cap \mathbb{R}$ is nonempty and bounded below, so this follows from Exercise 3.4.9.
    Case 3: $\beta = -\infty \notin A$. This follows from Exercise 10 applied to $A \cap \mathbb{R}$.

3.8 Limit Superior and Limit Inferior

1. (a) $(x_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}} = (-1,1,-1,1,\ldots)$. So $\limsup x_n = 1$ (the largest subsequential limit is 1), and $\liminf x_n = -1$ (the smallest subsequential limit is -1).

   (b) $(x_n)_{n\in\mathbb{N}} = (-1,2,-3,4,-5,6,\ldots)$. So $\limsup x_n = \infty$ $[(2,4,6,\ldots) \to \infty]$ and $\liminf x_n = -\infty$ $[(-1,-3,-5,\ldots) \to -\infty]$.

   (c) Since $1 + \dfrac{1}{n} \to 1$, there are 2 subsequential limits, namely 1 (n even) and -1 (n odd). So $\limsup x_n = 1$ and $\liminf x_n = -1$.

   (d) Since $-1 \le \cos x \le 1\ \forall\ x \in \mathbb{R}$, no subsequential limit can be outside the interval $[-1,1]$. Since $(\cos 2n\pi)_{n\in\mathbb{N}}$ converges to 1 and $(\cos(2n+1)\pi)_{n\in\mathbb{N}}$ converges to -1, $\limsup\cos(n\pi) = 1$ and $\liminf\cos(n\pi) = -1$.

   (e) $\left(\sin\dfrac{n\pi}{2}\right)_{n\in\mathbb{N}} = (1,0,-1,0,1,0,-1,0,\ldots)$ and $1 + \dfrac{1}{n} \to 1$. So $\limsup x_n = 1$ and $\liminf x_n = -1$.

   (f) Since $e^{-n} \to 0$, $\limsup x_n = \liminf x_n = 0$.

2. Any sequence with limit ∞, for example $(n)_{n\in\mathbb{N}}$. That these are the only types of sequences in $\mathbb{R}$ with limit inferior ∞ follows from Propositions 3.7 and 3.6 part 1.

3. Let $\beta = \liminf y_n$. If $\beta = \infty$, the result is clear; so assume $\beta < \infty$. Let $(y_{n_k})_{k=1}^{\infty}$ be a subsequence of $(y_n)_{n\in\mathbb{N}}$ with $y_{n_k} \to \beta$. The corresponding subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n\in\mathbb{N}}$ has a subsequence with limit in $\mathbb{R}^{\#}$. Since $x_{n_k} \le y_{n_k}\ \forall k$, this limit must be $\le \beta$. Therefore, $(x_n)_{n\in\mathbb{N}}$ has a subsequential limit which is at most β, and so $\liminf x_n \le \beta$ (because $\liminf x_n$ is the smallest subsequential limit of $(x_n)_{n\in\mathbb{N}}$).
   Let $\alpha = \limsup x_n$. If $\alpha = -\infty$, the result is clear; so assume $\alpha > -\infty$. Let $(x_{n_k})_{k=1}^{\infty}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$ with $x_{n_k} \to \alpha$. The corresponding subsequence $(y_{n_k})_{k=1}^{\infty}$ of $(y_n)_{n\in\mathbb{N}}$ has a subsequence with limit in $\mathbb{R}^{\#}$. Since $x_{n_k} \le y_{n_k}\ \forall k$, this limit must be $\ge \alpha$, and so $\alpha \le \limsup y_n$ (because $\limsup y_n$ is the largest subsequential limit of $(y_n)_{n\in\mathbb{N}}$).

4. The argument below mimics the proof of Proposition 3.8. Let $\alpha = \liminf a_n, \beta = \liminf b_n$, and $\gamma = \liminf(a_n + b_n)$. Since we want to show $\alpha + \beta \le \gamma$, we may assume $\gamma \ne \infty$ and $\alpha + \beta \ne -\infty$.
   Since γ is a subsequential limit of $(a_n + b_n)_{n\in\mathbb{N}}$, $\exists$ a subsequence $(a_{n_k} +$

$b_{n_k})_{k=1}^{\infty}$ of $(a_n + b_n)_{n\in\mathbb{N}}$ with $a_{n_k} + b_{n_k} \underset{k}{\rightarrow} \gamma$. The sequence $(a_{n_k})_{k=1}^{\infty}$ has a subsequence $(a_{n_{k_j}})_{j=1}^{\infty}$ with limit u in $\mathbb{R}^{\#}$ where $u > -\infty$ since $\alpha \neq -\infty$, and the sequence $(b_{n_k})_{k=1}^{\infty}$ has a subsequence $(b_{n_{k_j}})_{j=1}^{\infty}$ with limit v in $\mathbb{R}^{\#}$ where $v > -\infty$ since $\beta \neq -\infty$. Then $(a_{n_{k_j}} + b_{n_{k_j}}) \underset{j}{\rightarrow} \gamma$ and $(a_{n_{k_j}} + b_{n_{k_j}}) \underset{j}{\rightarrow} u+v$. By the uniqueness of limits, $\gamma = u+v$. Since α and β are the smallest subsequential limits of $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ respectively, $\alpha \leq u$ and $\beta \leq v$. Therefore, $\alpha + \beta \leq u + v = \gamma$.

5. Let $\varepsilon > 0$. Since $x_n \rightarrow x \in \mathbb{R}$, $\exists\ n_0 \in \mathbb{N}$ such that $n \geq n_0 + 1 \Rightarrow |x_n - x| < \varepsilon$. (We are using $n_0 + 1$ instead of n_0 for notational convenience below.) Following the hint, $n \geq n_0 + 1 \Rightarrow$

$$|y_n - x| \leq \frac{|x_1 - x| + |x_2 - x| + \cdots + |x_{n_0} - x|}{n} + \left(\frac{n - n_0}{n}\right)\varepsilon.$$

By Proposition 3.8,

$$\begin{aligned}
0 &\leq \limsup |y_n - x| \\
&\leq \limsup \frac{|x_1 - x| + |x_2 - x| + \cdots + |x_{n_0} - x|}{n} + \limsup \left(\frac{n - n_0}{n}\right)\varepsilon \\
&= 0 + \lim_{n\to\infty} \left(\frac{n - n_0}{n}\right)\varepsilon \\
&= \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $0 \leq \liminf |y_n - x| \leq \limsup |y_n - x| = 0$; hence, $\lim_{n\to\infty} |y_n - x| = 0$ by Proposition 3.7. Therefore, $\lim_{n\to\infty} (y_n - x) = 0$ and so $y_n \rightarrow x$.

6. Let $(x_n)_{n\in\mathbb{N}} = (0, 1, 0, 1, \ldots)$. Since

$$y_n = \begin{cases} \dfrac{1}{2} & \text{if } n \text{ is even} \\[2ex] \dfrac{\frac{n-1}{2}}{n} = \dfrac{n-1}{2n} & \text{if } n \text{ is odd,} \end{cases}$$

$y_n \rightarrow \dfrac{1}{2}$.